The Barber Paradox, Russell’s Paradox, And Some Suggestions For An Alternative To Set Theory

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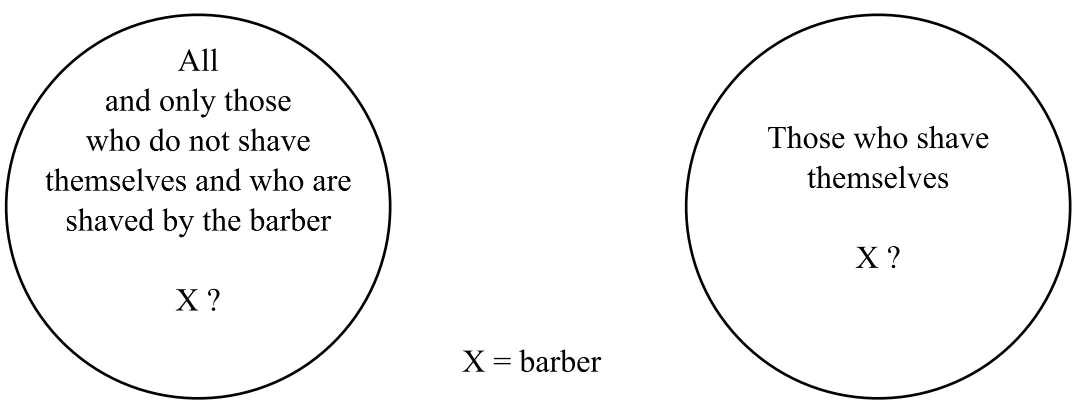
Barber Paradox

Here is the Barber Paradox, as stated by Bertrand Russell: You can define the barber as ‘one who shaves all those, and those only, who do not shave themselves’.

Does the barber shave himself? He cannot because he can only shave those who do not shave themselves. But if he does not shave himself then he would have to be included in the class of people who are shaved by the barber, or so it appears, because Russell said that the barber shaves ‘all those who do not shave themselves’. But if he is shaved by the barber then he is shaved by himself. It seems as though we contradict ourselves no matter how we answer, which is why it is considered a paradox. Here is a way to visualize it:



Here we have two classes. The ‘X’ represents the barber, and the problem is that we do not know which class to put him in. Either way leads to a self-contradiction.

There have been some proposed solutions. W.V. Quine claimed that there is no such barber. Albert Wang argues that the correct solution is: ‘The barber should shave himself and should only shave himself once.’[1] Wang believes that the barber should shave himself once because if he

[1] Albert Wang, “The Inequivalence of the Barber's Paradox and Russell's Paradox and the Solution to the Barber's Paradox”, International Journal of Multidisciplinary Research and Modern Education, Volume 6, Issue 2, Page Number 41-47, 2020.

has never shaved before then at that point he is a person who does not shave himself (not yet anyway) therefore he should be shaved by the barber. After he has shaved he should never shave again because from then on he is a person who shaves himself.

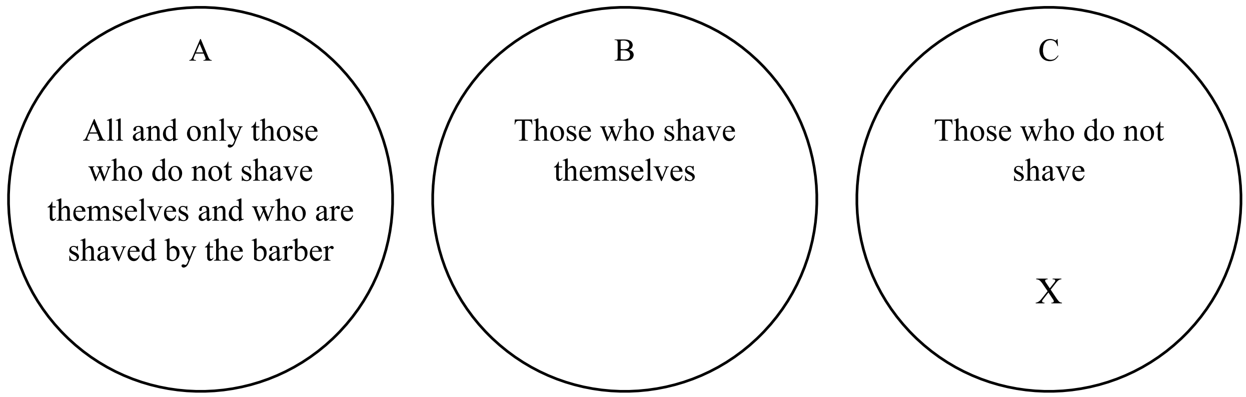
I see a couple of problems with Wang's solution. The first is that there is a difference between one who shaves himself currently, or on a consistent basis, and one who has shaved himself once before in the past. If the barber has only shaved once in his life, and that was twenty years ago, should he really be classified as 'one who shaves himself'? Certainly he does not do so currently, as Wang acknowledges. But that is just a minor issue. The real problem is that even if the barber has never shaved before it would still result in a self-contradiction for him to do it, even the first time that he shaves. Who is doing the shaving, himself or the barber? Since he is the barber it has to be both, so we still have a self-contradiction. Being shaved by the barber, according to the definition, necessarily entails not shaving yourself, thus we would be saying that he is both shaving himself and not shaving himself. Another way to think about it would be to ask ourselves how we would classify him after he has shaved. Wang says it should be as a person who shaves himself, but that is not necessarily the case. We could just as easily say he was shaved by the barber, even if it was only once. You could not put the barber in the class of those who are shaved by the barber because he shaved himself, but you also could not put him in the class of those who shave themselves because he was shaved by the barber, and the condition for being shaved by the barber was that those who are shaved by the barber are those who do not shave themselves.

Wang also mentions that others have suggested that maybe the barber is a woman and would therefore not need to shave. I think that answer is the closest to the true solution, but we could make it more general.

I believe that the best answer is simply to say that the barber does not shave. The reason could be that the barber is a woman, but it could also be a male barber with a long beard who chooses not to shave. That might be a somewhat ironic fashion choice considering his profession, but it is certainly logically possible. Thus, Quine's proposed solution is incorrect because there definitely could be such a barber.

If the barber does not shave then he/she does not belong to either class. There would be a third class of people - the group that does not shave - which would include most women, 6-month-old babies, 7-year-old children, and most 14-year-old boys, among others. Within this group there could also be men who choose not to shave. Because of how the other classes were defined it leads to a self-contradiction to try to put the barber into either one of them, but it does not result in a self-contradiction to put him (or her) into this one.

So really we have three classes:



X= barber

Of course this answer depends on how literally we take the phrase ‘all those who do not shave themselves’; if that means that everybody must be shaved by someone, either the barber or themselves, then my answer would not work. But it seems absurd to think that the barber would be shaving the faces of 9-month-old babies and 4-year-old children; if they do not need to shave then they would not shave themselves or be shaved by the barber. In addition to being totally unnecessary, and therefore a waste of time, it would cost money to have them be shaved by the barber and even to shave themselves (though probably less) which could be spent on other things. (Some members of the class would not even be able to shave themselves, such as babies and young children.) So it seems like we should assume that ‘all and only those who do not shave themselves’ refers only to the people who actually need and/or want a shave but do not shave themselves.

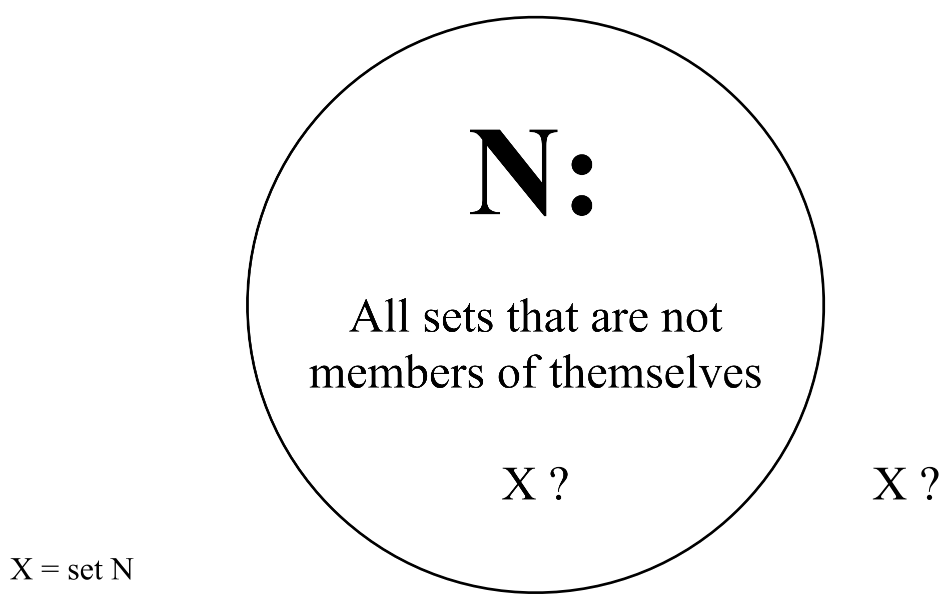
Before leaving this section there is another issue that should be mentioned. Let’s suppose that the barber’s name is Joe. When and if he were to shave himself, would he be doing it as an ordinary Joe or Joe the Barber? If an ER doctor diagnosed himself at his home and wrote out a prescription for himself, would we really say that he ‘went to the doctor’? If a doctor ‘goes to the doctor’ it is assumed that we are talking about some other doctor, not himself. If a chef cooks a meal at home for herself and no one else, would we really say that the meal was cooked by a chef? I think that would be strange because it is not in her role as a chef that she cooked the meal. In what capacity is the chef acting, as a professional chef or an at-home cook? If Joe shaved in front of his bathroom mirror at home, like I do, I wonder if it is really correct to say that he was ‘shaved by the barber’ even if he is a barber in other contexts. Typically getting a haircut or a shave from a barber would imply that someone else did it for you in exchange for money, even if you also have your license and do it for other people as your profession. Will the barber also bill and pay himself for the service? He must be one quirky barber if he does. No ordinary Joe, that’s for sure.

Russell’s Paradox

Now we will move on to Russell’s Paradox. There are some who regard the Barber Paradox as a version of Russell’s Paradox that is easier to understand. I am not one of them. There are some similarities, but I do not consider them to be equivalent for reasons that will become apparent as we discuss Russell’s Paradox and Set Theory.

For those who are not familiar with Russell’s Paradox, it is a basic tenet of Set Theory that sets or classes can be formed merely by describing the properties of their members. So the paradox is this: suppose that we have the set of all sets that are not members of themselves; is this set a member of itself? If it is then it must have the characteristic (or property) of not being a member of itself, but on the other hand, if it is not a member of itself then it must be included in the set. Attempting to answer either way leads to a self-contradiction, which is why it is considered a paradox.

This may be a little bit hard to follow if it is your first encounter with it. Perhaps having something to help visualize it would make it easier to understand. Let’s call the set of all sets that are not members of themselves ‘N’:



The question is whether N is a member of this set (itself). According to how the set is defined, N would be included in the set if it has the property of not being a member of itself. So if it is a member of itself then it has to be a set that is not a member of itself, an obvious self-contradiction. But if it is not a member of itself then it has the property that defines the set so it would be a member of N. So if it is not a member of itself then it is a member of itself. Once again a self-contradiction. So is N a member of itself or not? No matter how you answer it seems to lead to a self-contradiction.

My intuition is that it is absurd to say that any set can be a member of itself. The set is the whole, and subclasses and members are like parts of the whole; nothing could be both the whole and simultaneously a part of that same whole because that would mean that it is greater than (or less than) itself, which is absurd.

There are other absurdities. Suppose that we have a set, we will call it x, which has 4 members, A, B, C, and D, as well as itself as a member.

Set: x
Members of x: A, B, C, D, x, or 'itself'

Members of x: A, B, C, D, x, or 'itself'

Members of x: A, B, C, D, x, or itself

. . . ad infinitum . . .

By having x as a member of itself we are essentially double counting all of the members of x; and, because the member class must be identical with the class, the member class would also have to include itself as a member, and that member would include itself as well, and so on.

According to the principle of substitution if two things are equal then one may be substituted for the other. So if set x = A, B, C, D, x then the latter may be substituted for set x, and we end up with not 5 members of the set, but an infinite number. But they are not different members we are just recounting the same members over and over, ad infinitum. Why?

If x is considered complete without including itself, then it is complete, so no other members should be included in it. Thus x does not include itself. But if we say that x is not complete without including itself then how would adding itself make the set complete? After all, it is just a repetition of the same thing, so if x was incomplete before, and the member set that is added is just an exact duplicate of this, then how is adding that going to make the set complete? If it was incomplete before then what is added is going to be lacking in the exact same way.

Genus, Species, and Difference

Biological classification, which is based upon Aristotle's work, is a good example of how classification ought to be. Living things are organized according to domain, kingdom, phylum, class, order, family, genus, and species. This organization goes from most general to most specific, with each subclass representing a more refined distinction that does not include as many members as the prior class.

One can use a similar classification system to organize other things, not just plants and animals, and Aristotle did. In fact he sometimes defined things based upon their genus, species, and difference, which is a good way of doing it if you already know some things about the genus and the difference.

For example 'ice' could be defined as 'frozen water'. 'Water' would be the genus, 'frozen' would be the specific difference, or what distinguishes it from other types of water, and 'ice' is the species.

As you can see, in logic the terms 'genus' and 'species' have a somewhat different meaning than they have in biology. Even though how they are used in biology is actually based upon Aristotle's ideas he used the terms much more broadly. In logic 'genus' simply refers to the larger or more broadly defined class, and 'species' refers to a subclass of the genus. Thus, in logic we could speak of the genus 'animals' and the species 'mammal', or the genus 'human' and the species 'female', or the genus 'chair' and the species 'rocking chair', etc.

The genus is always more general than the species. The specific difference narrows the genus class, restricting it in some way to only some members of the genus. The specific difference explains how that species is different from other species of that genus, of which there could be many. (It is necessarily implied that each genus must have more than one species.) Just think of all the different types of chairs there are; each of those types would be a species of the genus chairs.

If a class was a member of itself (or subclass of itself) it would have to simultaneously be both genus and species. But how could it be a subclass or specific type of the genus when it has exactly the same members as the genus? The species is supposed to be more specific than the genus, a type of the genus, not an exact duplicate of the genus. But it would have to be an exact duplicate or it would not be 'itself'. But if the species is 'itself' how could there be any other species? Polar bears and black bears are both species of the class bears, and differentiated from each other; it makes sense to say that black bears are a specific type of bear, or that polar bears are a type of bear, but how could bears be a specific type of bear? It makes no sense. Ice can be a type of water but water is not a type of water.

There could not be a specific difference or the genus and species would not be identical; but if they are identical what is the point of considering it to be two separate classes? We are not going from the most general classification to the most specific, we are just repeating the exact same class for no good reason. And since the classes have to be identical, if the genus class includes itself as a member then the species class would have to as well, and so on, ad infinitum.

If one were to add up all the members of all of the subclasses of bears it would equal the same number of members that the class of bears has, because the parts equal the whole. But if a set is a member of itself then the species, or the parts, add up to more than the whole because one of those species is equivalent to the whole, or the genus, itself, and yet there must be other parts in addition to that or it would not be a species.

Arthur C. Clark, in a letter to Clifford Pickover, published in one of Pickover's books *Black Holes: A Traveler's Guide*, said that if there are infinitely many digits in pi then pi must contain every possible series of numbers. Therefore it contains pi, or in other words all of the digits in pi. And that subset, also having an infinite number of digits, must also contain pi, and so on forever . . . After explaining this Clark says: 'This way lies madness.'

I agree, it is madness. How could pi contain all the digits of pi and additional digits besides? If there are additional digits then you must not have had 'all the digits of pi' to begin with (for the subset). On the other hand, if both the set and the subset are identical, meaning that they include and exclude exactly the same digits, then there is no reason to consider them to be separate sets. This is absurd not because it leads to an infinite regress of sets and subsets: infinite sets can have infinite subsets and potentially an unlimited amount of subsets. The set of integers, for example, could be divided in an infinite number of ways, so it could have an infinite number of subsets. The reason that it is absurd is because you would be saying that the set contains itself as a subset, yet also has other subsets and members besides. It is true that pi contains 'all the digits of pi' by definition, but also by definition it could not include any other digits or else it would be something other than the set of digits in pi. Hence the set is not 'all the digits of pi' or the subset is not 'all the digits of pi'. This attempted organization and categorization of the digits does not make any sense. We really should just say that all of the digits are part of the same set.[2]

There is an old song called 'I am my own grandpa'; this whole discussion reminds me of that, except that the song was meant to be a joke. A set being a species or particular type of itself is even more absurd than saying that you are your own grandpa, it is more like saying that you are your own mother.

Instances in which a set appears to be a member of itself

And yet, having said all that, there do appear to be some instances in which a set is a member of itself. For instance, in William Lane Craig's book *The Kalam Cosmological Argument* he talked a little bit about Russell's Paradox and he gave this example: 'the set of all things mentioned in this chapter is itself mentioned in this chapter and so would seem to be a member of itself.' (Page 90.)

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[2] Another thing to consider is that one cannot ever complete an infinite series or a series without end (even if it has a beginning), so you could not ever really say that you have 'all the digits of pi' represented in the set anyway. The total number of digits would be forever growing without end, you would never have all of them listed.

A friend of mine provided this example: ‘the set of all sets that have more than two members’. Since this set has more than 2 members it seems like it has to be a member of itself.

I also thought of another one myself which is perhaps the simplest and most direct, and that is simply the set of all sets. Since it is also a set it seems like it has to be included.

I argued in a previous essay that examples like these last two would not be members of themselves, they would instead be members of a more general set. The ‘set of all sets’, rather than being a member of itself, would be a member of the ‘set of sets of all sets’. But as I have thought about it more I have come to realize that even if it was a member of that set it would also have to be a member of itself as well unless I want to say that it is not a set. (Or so it appears.) That is the potentially the biggest problem for my position: I might think that it is self-contradictory and makes no sense to say that a set is a member of itself, but if it is not then it seems as though I am committing myself to saying that it is not a set, and that would be even more ridiculous. Similarly, I cannot really argue that the ‘set of all sets that have more than two members’ has less than two members itself, or two members exactly; obviously that could not be right, so how could it not be included in the set if it has more than two members?

I will get to answering these objections eventually, but first we need to do some preparatory work.

Categorical Logic

To aid us in further analysis I would now like to turn to categorical logic. Set Theory ought to be in harmony with categorical logic because they are both related to classes.

Let’s first look at an ordinary categorical proposition:

‘All cats are mammals’

‘All’ is the quantifier, which tells us how many members of the subject class are claimed to have the predicate, ‘cats’ is the subject, or the item or class that the claim is about, ‘are’ is the copula, and ‘mammals’ is the predicate. The predicate can be a characteristic or property that the subject either has or lacks, and/or another class that it would either belong to or not belong to.

This particular categorical proposition is true because the predicate ‘mammals’ is the genus, which is more general, and the class of ‘cats’ is a species of that genus. Thus it is true that every member of the species must also be a member of the genus, since the genus includes all members of the species plus additional members besides. It is also true that no members of the species are not members of the genus because the species is a legitimate subset of the genus.

More generally we may say that if the subject is a legitimate subclass of the predicate class then it has to be true that all members of the subject class are also members of the predicate class. That is the connection between the classification system discussed above, with genus, species, and difference, and categorical logic. (Both ideas originally coming from Aristotle.)

But suppose that ‘mammals’ was a subclass of itself. ‘All mammals are mammals’ is a true statement, in fact it is a tautology, but it is also not really worth pointing out because it is so obvious. If a class was a member or subclass of itself then it would have to be both the species (subject) and the genus (predicate) in a categorical proposition, and that is very strange.

Let’s now return to some of the examples mentioned earlier of sets that are thought to be members of themselves. Let’s first look at ‘the set of all sets that have more than two members’. If it was a categorical proposition it would actually be this:

All sets identical to ‘the set of all sets that have more than two members’ are ‘the set of all sets that have more than two members’.
quantifier subject copula predicate

If it is really a member or subclass of ‘itself’ the subject class and the predicate class must match exactly. They do, but does this categorical proposition tell us anything new? To me it simply states the obvious, so it is needless redundancy.

Suppose that we say that the ‘set of all sets that have more than two members’ has A, B, C, D, itself, etc., as members. If we think about how this would work for another member of the set, A, which could stand for simply a class that has more than two members, such as dogs, or cats, or humans, etc., then this would be the categorical proposition:

All ‘A’ are ‘the set of all sets that have more than two members’.
subject predicate

or

‘A’ is ‘the set of all sets that have more than two members’.
subject predicate

This categorical proposition, in either form, would not be true. ‘A’ is defined as a set that has more than two members but it is not *the set of all sets that have more than two members*. For instance, if we are talking about the class of humans then it is true that the class has more than two members but false that it is the set of all sets that have more than two members. So is A even a member of this set? I do not believe that it would be, despite the fact that it is a set that has more than two members, because having more than two members is not the predicate that defines this set, the predicate is ‘the set of all sets that have more than two members’, and there could really only be one class that has that predicate. If the set is a member of itself it would have to be the only member. But why would we say that it is a member of itself if there are no other members? Why not just say that it is all one class?

But maybe you will object that I am not forming the predicate correctly; perhaps it should be 'sets that have more than two members' rather than 'the set of all sets that have more than two members'. Let's first consider what that would mean for 'A', an ordinary set that has more than two members, such as 'horses' or 'humans':

All 'A' are 'sets that have more than two members'.
subject predicate

or

'A' is a 'set that has more than two members'.
subject predicate

The first one may be technically correct but it seems strange to say it that way; the second version, however, makes perfectly good sense, and is both true and meaningful; indeed, it is the case that the set of horses (or the set of humans) is a set that has more than two members. The statement tells us that 'A' has this characteristic and that seems like it is something significant and worth pointing out.

But what if we were to instead say:

All 'sets that have more than two members' are 'sets that have more than two members'.

or

'Sets that have more than two members' is a 'set that has more than two members'.

Maybe the latter is meaningful and would be worth pointing out. It does tell us about a characteristic of the set, and this time it does seem like it at least could be true that the set has that characteristic. However it does not have to be true because it is not specified how many sets that have more than two members are included in this one. The set of dogs and the set of cats both have more than two members, what if these are the only sets that are included in 'sets that have more than two members'? If that was the case then the set would have exactly two members (the set of dogs and the set of cats) so it would not have this predicate and this categorical statement would not be true. Thus the set itself would not necessarily have to be included as a member.

Now, it is true that 'the set of all sets that have more than two members' would have more than two members. So this categorical statement would be true:

‘The set of all sets that have more than two members’ is a ‘set that has more than two members’.

But notice that the subject and the predicate do not match exactly, and they would have to if a set is a member of itself. The predicate is defined by a slightly different property than being ‘the set of all sets that have more than two members’. Therefore it is true that this set (the subject) has more than two members but false that it is a member of itself. It is actually a member of a closely related but different set than itself.

One could say something similar about Craig’s example of ‘the set of all things mentioned in this chapter’. If that is the predicate then no other things that are mentioned in the chapter would have that predicate. There is a difference between being *a thing* mentioned in the chapter and being *all things* mentioned in the chapter. But if the predicate is instead ‘things mentioned in this chapter’ would ‘the set of all things mentioned in this chapter’ be a thing mentioned in the chapter? Yes. Does that mean that is a member of itself? No, because the subject ‘all things mentioned in this chapter’ and the predicate ‘things mentioned in this chapter’ are different.[3]

Let’s now go back to Russell’s own example, ‘the set of all sets that are not members of themselves’. Which sets would be members of that one? He thought that it would be any set that is not a member of itself, but similar to what we have seen in previous examples, if that is the predicate then other sets that are not members of themselves, such as ‘mammals’, or ‘cups’, would not be members of that set because they do not have the predicate of being ‘all sets that are not members of themselves’. There could only be one set that has that predicate, itself. But then we would be saying ‘the set of all sets that are not members of themselves’ (subject) is ‘the set of all sets that are not members of themselves’ (predicate). What would be the point of stating the obvious?

But if the predicate is ‘sets that are not members of themselves’ rather than ‘all sets that are not members of themselves’ or ‘the set of all sets that are not members of themselves’ then it is not specified how many sets must be included in that, whether it is all or only some of them, so ‘itself’ would not have to be included.

Since ‘the set of all sets that are not members of themselves’ is not a member of itself either, one could state the true categorical proposition: ‘ “the set of all sets that are not members of themselves” (subject class) is “a set that is not a member of itself” (predicate)’ because it is true that it does have that characteristic, as do all other sets, but that does not mean that is a member of itself. It is not, as we already said.

[3] One could make essentially the same argument for ‘the set of all sets’; it is obviously a set, but it is not a member of itself, it is a member of ‘sets’, and ‘sets’ would not necessarily be a member of itself either because it is not specified whether it must include all sets.

Class Complements

There is another type of set that some might think of as an example of a set that is a member of itself. Class complements are classes that are based upon a characteristic that members of the class lack. Because of that I refer to them as anti-predicates. For instance 'non-dogs' would be a class that includes everything that is not a dog, such as cats, humans, horses, tables, etc. Anything that is not a dog would be included in it. Is this class a member of itself? It is not a dog, after all, so some may argue that it would be.

It is not the case that this class (nor any other) is a dog, but I do not think that it should be thought of as a non-dog either. The truth is that it neither has nor lacks the characteristic of being a dog because it is not an object, it is just a collection. Since the collection is not a separate thing it is not a thing that must be included in the collection in order for the collection to be complete.

Should the class of dogs be included in the class of 'non-dogs'? I do not mean the individual members of the class, I mean the class itself; that class is not a dog either. Do we really want to say that 'All dogs are non-dogs' is true simply because the class of dogs is not itself a dog? How about 'All terriers are non-dogs', or 'All German Shepherds are non-dogs'?[4] How can the set of dogs, and the set of German Shepherds, and the set of terriers be included in the set of non-dogs if the members of those sets are not included in it?

Now I guess someone arguing for the other side could say that this is confusing merely because we are not making proper distinctions. If you were to say that the class of dogs is not itself a dog, or that the class of terriers is not itself a dog, then there is no problem. I could not disagree. But why are we even talking about the characteristics of the class itself? When we group objects together based upon a shared characteristic or a characteristic that they lack (predicate or anti-predicate) we are referring to the characteristics of the objects, not to the characteristics of the 'grouping' of those objects, as though that collection is itself an object that has its own predicates that are separate from the predicates of its members. The group of objects is not a separate object.

Bertrand Russell would have disagreed with me on this. He said in a book titled *Introduction to Mathematical Philosophy* (1919) that he originally thought of what has come to be known as Russell's Paradox in 1901 while analyzing a mathematical proof by Georg Cantor. It was after considering a supposed 'class of all imaginable objects' that he thought of the potential contradiction:

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[4] You could do the same thing with other classes too. For example the set of even numbers is not itself an even number, so would it be true to say 'All "even numbers" are non-even numbers'? I say no because it is not an odd number either, it is not a number at all, so neither 'even number' nor 'non-even number' applies. One could say 'All even numbers are non-numbers' or '"Even numbers" are non-numbers' because the class itself is not a number, but the problem is with treating the class as though it is a object that has or lacks the predicate.

> The comprehensive class we are considering, which is to embrace everything, must embrace itself as one of its members. In other words, if there is such a thing as "everything," then, "everything" is something, and is a member of the class "everything."

He then goes on to state Russell's Paradox. In this quote he makes it seem as though if 'everything' was not included in the class 'everything' that the class would be incomplete because an object that should have been included was left out. He says that '"everything" is something' but actually it is not. The term 'everything' does not refer to a particular 'thing', it refers to the entire collection. The pile of stuff that is 'everything' is not also an additional item within the pile.

By Russell's reasoning here 'non-everything' would also have to be included in the set 'everything' because 'non-everything' would be a set, even if it is empty, and if sets count as things (or 'something') then that must also be included. There are all sorts of absurdities here.

One could make a similar point using prior examples. For instance, if 'sets' includes itself as a member because it is a set then it would also have to include 'non-sets' as a member because, ironically enough, 'non-sets' would be a set. Thus it would be true that all non-sets (subject) are sets (predicate) or at least that the set of non-sets is a set, even though its members are not. So we include the set but we do not include its members?

The set of integers is not itself an integer, but it would be confusing and self-contradictory to say 'All integers are non-integers'. It makes perfect sense, however, to say 'All cats are non-integers'. The reason is not because the class is not an integer (even though that is the case) it is because the members of the class - the cats - are not integers. In other words we are talking about the characteristics of the members, not the characteristics of the class itself. The class is neither an integer nor a non-integer, it is not a subject that has predicates or anti-predicates. When we talk about 'integers' we mean the members of the set not the set itself. Thus, 'All integers are non-integers' is false because that is certainly not a characteristic of the members of the set.

The source of the confusion is the ambiguity that results from thinking of the set as having a different predicate than what its members have, or just thinking of the set as a separate object that is either included or excluded from itself. The set is just a collection, not an independent 'thing'.

There is a fundamental distinction between sets and members which is that sets are groups whereas a member is a single item. However, it is true that the set can sometimes be treated as a single unit. For instance, a baseball team could be part of a league; in that case the team is a member and the league is the class. In some contexts the team may mean the collection of players that are on it but in other contexts it may mean the franchise, or the brand, or just a member of the league of teams. When we mean it as a franchise it would be thought of as a single entity and in those cases it does seem to be a 'thing' that is either included or excluded from the league because we are talking about the organization itself, not just the collection of players. The same would be true if you are referring to a corporation or a government. In some contexts it would not be just a collection of employees, it would be an entity itself. Even with the human body you could think of it as a collection of cells or as a single unit. But the key point is that it cannot be both a collection of units or members and a member of that same collection; a team can be a member of a league of teams, but it cannot be a player on the team (itself). A division of four teams also would not be a unit of the league (because it is a group of units), it is a subclass. You have to pick which way the set is being referred to, either as a single unit, which could be a member of some other class, or as a collection of members, and stick with that throughout. We cannot blend them together within the same context without it resulting in ambiguity and confusion. In both the prior diagrams and in the ones that follow I use a circle to represent a class and an X to represent a member. One way of putting this point would be to say that the class must be represented either with a circle or an X, it cannot be both at once, or both within the same diagram. One will notice that the diagram on page 4 for set 'N' is confusing because of this. That diagram is not correct, but it does accurately depict Russell's Paradox, which says something important about Russell's Paradox.

Replacing Set Theory

I will now be discussing in a more general way what I consider to be some of the underlying problems with Set Theory and then give some suggestions for how we could create an alternative theory that is better. I am not exactly sure what to call it though, 'Elements Theory' perhaps? Does it really even need a name? I am not sure that it does. To me these are just some common sense stipulations and clarifications regarding classes, so one could think of it as a natural outgrowth of my version of categorical logic. Technically this is actually more foundational, so I suppose one could see categorical logic as being an outgrowth of it, but conceptually that is not the case. We usually start by analyzing classes that are well-defined and for which the answers are not controversial, then eventually we look at instances in which the answers are not as clear, and that is where this would fit in.

Let's start with the most basic, which is how to define a set. A set is a collection or group of things. This definition is widely used in both logic and math,[5] and yet the logical implications of it are not being recognized in Set Theory. Specifically, this definition necessarily implies that there has to be 'things', or elements, if there is a set, and there has to be more than one of them. Let's start with that latter point.

[5] For instance this is the definition given by Richard Hammack in his online math textbook called *Book of Proof*. As far as I can tell his explanations and the views presented in this textbook represent the current consensus view in mathematics pretty well, which is what a good textbook ought to do.

One is a lonely number

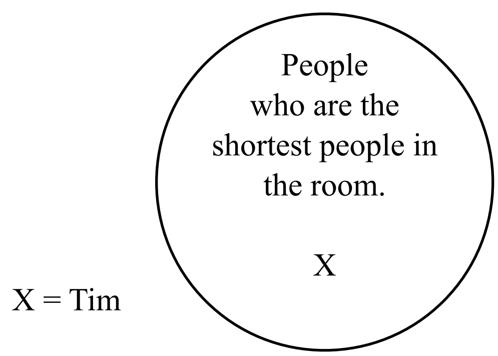
A set must have more than one element/member, and there must be some characteristic (at least one) that is common to both or all of them. The fact that the members have this shared characteristic is why we group them together. For example, people who are in the same profession, such as accountants, lawyers, doctors, etc.

If it is a single item then it is not a group, therefore it is not a class. At best a single item is sort of a quasi-class. I suppose it could be the beginning of a new class, the first of something, but it is not really a class until there is a group. It would be best to just consider it a single item and leave it at that until there are more of them.

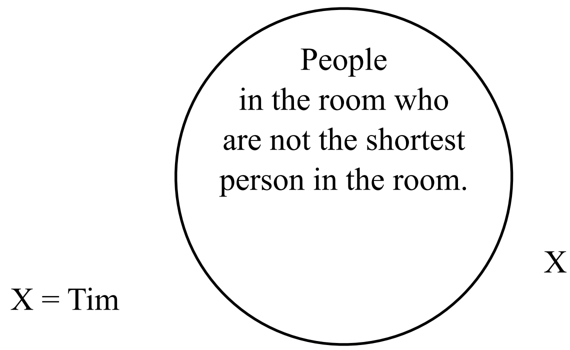
Sometimes in categorical logic a single object is treated as a category. For example, 'All things identical to the Queen of England', or 'All things identical to China' are considered categories that have only one member. This is primarily done for convenience: it allows us to use Venn diagrams and other tools and techniques typically used in categorical logic to check the argument for validity. The idea is that this allows for a wider range of arguments to be analyzed. But it is far from a perfect fit. It is really awkward phrasing first of all, but the bigger problem is that this is not consistent with the definition of a class/set/category as a group.

I think there are better ways of doing it. I do not diagram individual objects as sets in my version of categorical logic. I use a circle to represent a class, and an X to represent a single thing.

However, this does raise a question regarding how one should diagram a statement such as 'Tim is the shortest person in the room'. We could perhaps do it this way:

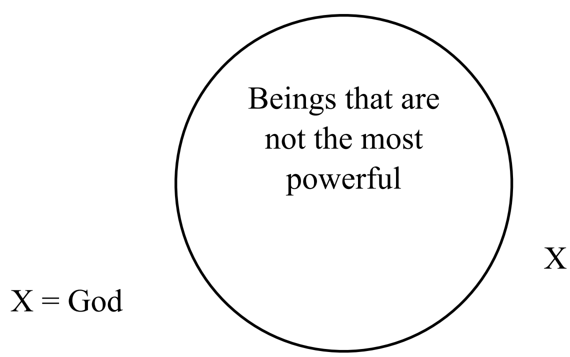


But the problem is that we are still representing a single individual (the shortest person in the room) as a class, and saying that Tim is in it, which is far from ideal. I think this is a better way to diagram it:



Perhaps the phrasing is still a little clunky; we could just say 'people in the room who are not the shortest'. I would say that in ordinary conversation but then diagram it using this phrase just to make certain that we are being absolutely clear. In this case we have a class of people and Tim is excluded from that class. This enables us to keep how we diagram a class separate from how we diagram an individual object. This does necessarily imply that there is more than one unnamed person in the class 'people in the room who are not the shortest' but I do not think that is a problem because when we say 'shortest' that itself implies that there is more than one additional person in the room besides Tim, for if there were only two people, Tim and one other person, then we would, or should, just say that Tim is shorter. In that case the other person would not be a class, nor would Tim, it would just be two individual things. I do not think there would be a diagram for that using categorical logic, since neither person is a category. You would need to find some other way to represent it, maybe using propositional logic, since it is a statement that is true or false assuming that Tim and the other person actually exist. (If they do not then the statement is undefined in actuality though it could have a hypothetical truth value.) At any rate, it would be outside the scope of categorical logic, and I think that we should be wary of trying to force categorical logic to apply when it does not naturally apply.

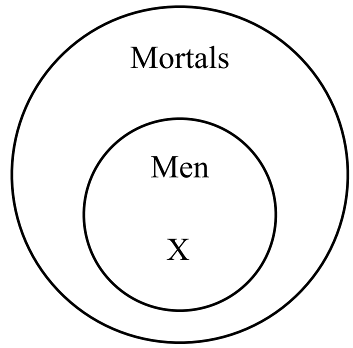
Here is another example: 'God is the most powerful being'



One might also say 'things that are not the most powerful' if preferred. If the statement is true then everything except God would be in that class, or if the more restricted class 'beings' is used then all beings except God.

The classic example of a valid argument is this:

All men are mortal
Socrates is a man
Therefore Socrates is mortal

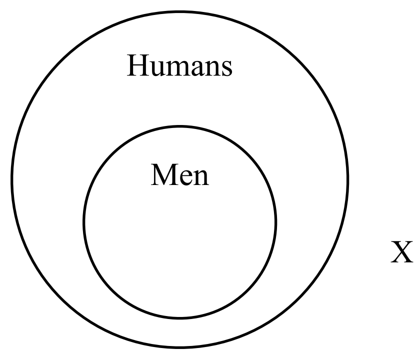


X = Socrates

According to the premises, the class of men is a subclass of the class of mortals, and Socrates is a member of the class of men; therefore it follows that he must also be a member of, or included in the class of mortals, which is why the argument is valid.

Here is another example:

All men are humans
Secretariat was not a human
Therefore Secretariat was not a man



X = Secretariat

This argument is also valid. If Secretariat (the famous race horse) was not a member of the class of humans then he could not have been included in the class of men, since the class of men is a subclass of the class of humans.

To me this is a clearer and better way to diagram the premises than if we were to consider Socrates or Secretariat to be classes.

A thing should be classified simply as an individual object if it is the only thing that has that predicate. No further classification than that is necessary or warranted. The object may belong to classes based upon other predicates that it has: for instance, the Queen of England, Queen Elizabeth, belongs to the class of humans, but when it comes to being the current queen of England she is in a class by herself, which really means that there is not a class, she is the only thing that has that predicate. She would, however, belong to the class ‘queens of England’ or the more general class ‘queens’ along with queens from past generations. The bottom line is that unless you are using the plural form it is not really a class.

The terminology of categorical logic presupposes that classes have more than one member. ‘Some S are P’ assumes that part of the S class has the predicate and part of it does not, which requires at least 2 members.[6] ‘All S are P’, or ‘Every S is a P’, which was the way that Aristotle originally stated it, also presumes that there is more than one member of S; we would just say ‘S is P’ if S were singular, there would be no need for a quantifier. Even ‘No S are P’ takes for granted that there is more than one S, or you would just say S is not a P.

Sets are not members and vice versa

The fundamental difference between sets and elements is that sets are groups whereas elements are singular. If sets must have at least two members (and usually it would be more) then this alone means that a set could not simultaneously be both the group and a member (singular) of the same group. In Russell’s Paradox the question is whether the set is a member of itself. If we were to say that it is that would be equivalent to saying that the set is both a single item that is part of the collection and also simultaneously the entire collection. That makes no sense.

Another reason that this distinction is important is that in Russell’s Paradox we are asked whether the set has the predicate that would give it membership in the class. But a collection is not a separate or additional ‘thing’ that has its own predicate. Only the elements/members, or in other words single units have or lack the predicate. When we talk about a class, such as dogs, having a predicate or characteristic, we do not mean the class (group) itself, we mean the dogs. The class itself is not a member or a non-member of the class, neither of those apply because it is not an object.

A set or class can sometimes be considered a unit, in which case it is thought of as being singular in that context, and it could be a member of a different class (a grouping of classes) but it could not be a unit of itself. The set has to either be a unit or a group of units. It cannot be both at once. Referring to it in an ambiguous unclear way is what leads to self-contradictions.

I could be wrong about this, perhaps there are exceptions that I am not aware of, but I also do not think that one can legitimately mix single items with sets and treat them all as members of the same set. For instance Richard Hammack, in *Book of Proof* gives as an example: E = {1,{2,3}, {2,4}}, which, he says, contains three elements, the number 1, the set {2,3}, and the set {2,4}. I think you would need to decide how the sets {2,3} and {2,4} are to be treated, either as sets or as units. If they are considered units then I do not see how the number 1 could also be a unit; it seems to me that if the elements of E are sets then all of its elements would have to be sets. On the other hand, if {2,3} and {2,4} are treated as sets, meaning a group of elements, then the

[6] For other interpretations of categorical logic ‘some’ means at least one, but in mine ‘some’ necessarily implies ‘some are not’, and vice versa, so the category would need to have at least 2 members. If the category did have only two members and the particular statement is true then one member would have the predicate and the other would not.

elements of set E are 1, 2, 3, and 4; {2,3} and {2,4} would be subsets of E, not elements of E. The number 2 is common to both, but that is okay, sometimes subsets overlap.

No class can be a subclass of itself

But even if a class could not be a member of itself we might still consider whether it could be a subclass of itself. Suppose A and B are sets. In Set Theory, if every element of A is also an element of B then A is considered a subset of B. I disagree with this definition because what if they actually have every element in common? For instance, suppose that A = {1, 2, 3}, and B = {1, 2, 3}.[7] According to this definition of subset, A is a subset of B because it is the case that every element of A is also an element of B and no elements of A are not elements of B. But one could also say that B is a subset of A because it is true that every element of B is also an element of A and no elements of B are not elements of A.

It does not make sense to refer to A (or B) as a 'subset' if both of them have exactly the same members. It is actually the same set (group of members), which is being referred to by a different name, or in this case a different letter, but it is still the same group of elements.

If we think of it in terms of species and genus, the genus is a more general classification and the species is a finer classification within the genus in which some difference is noted that would distinguish that particular species or type from other species of that genus. The members of the genus are grouped together based upon a similar characteristic but they are also separated and further classified based upon a difference or differences. So it is true that every element of A must also be an element of B, if A is a subset of B, but it must also be the case that B has at least one element that A does not have.

My definition of subsets would be this: If every element of A is also an element of B and some but not all elements of B are also elements of A, then, and only then, A is a subset of B. This is very close to how proper subsets are defined.

It is self-contradictory to say that A is a subset of B and also that B is a subset of A. They cannot both be the subset (or species) at once.

If a set were a subset of itself what would differentiate the species form from the other species of that genus (itself)? The genus could not have at least one member that is not included in the species. All the members would have to be identical or it would not be a subset of itself.

Two classes are identical if you could reverse the subject and the predicate and the resulting statement would be equivalent to the original. For instance, 'All domesticated dogs are members

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[7] In section 1.4 of *Book of Proof,* on powersets, Hammack lists both the empty set and an exact repetition of the set as being subsets of the set.

of Canis lupus familiaris' and 'All members of Canis lupus familiaris are domesticated dogs'. Both classes have exactly the same members and exclude all the same members so they are the same class identified by a different name. If we considered them to be different classes in our classification system it would really foul things up. There is an even bigger problem if we say that a class is a subclass of itself. We would not be moving from general to specific because the species would have exactly the same members as the genus. If it was a subclass of itself the classes would not even be called by a different name, it would even be referred to in the exact same way. So why consider them to be two separate classes? And how could one be a species of the other when there could be no specific difference? Saying that a class is a subclass of itself would be like saying that a slice of the pie is equal to the whole pie. A part does not equal the whole.

Some say that a part can equal the whole if the sets are infinite. However it is not the case that the sets are equal in the sense that they have the same cardinal number of members or units, such as if both had 17 members or two lengths were both 17 cm long. Only finite sets have a specific cardinal number of elements, so only finite sets could be equal in that sense.

The set of odd integers would be a subset of all integers. It is not the case that the set of odd integers has exactly half as many members (cardinal number) as the set of all integers, as it would be if they were finite sets; but contrary to what you may have heard, or what often seems to be implied, it is also not really correct to say that both sets have exactly the same amount of members (cardinal number) either because both sets are unlimited in how many members they can have, so neither one can be measured or quantified. They are equal only in the sense that both of them are unmeasurable. But it is still the case that the genus is more broadly defined, meaning that it must include at least one member that is not included in the species. That obviously would not be the case if the set was a species of itself.

'The empty set' is not really a set

Another very strange idea that comes from math is that the empty set is said to be a subset of every set. The reason given for why this is the case is that one way of defining how set A would not be a subset of B is if there is at least one element of A that is not an element of B; this is, of course, the Aristotelian contradictory of the claim that all members of A are members of B. Since the empty set has no elements at all it would always be the case that it has no elements outside of any set that you are referring to. Thus, according to this reasoning, it must be a member of every set. This is similar to John Venn's reasoning regarding universal categorical statements; he thought that both 'All S are P' and 'No S are P' are always true if the subject class is empty. However, it is bad reasoning. It leads to many self-contradictions.

The definition of a subset is not applicable to the empty set. If you were to say 'All elements of the empty set are also elements of B' that claim is not true or false, it is undefined. Because there are no elements of the empty set (and because there are no elements it is not even really a set) the statement cannot have an actual truth value. If this is undefined and does not have a truth value then its Aristotelian contradictory ‘Some elements of the empty set are not elements of B’ also does not have a truth value. None of the categorical statements have a truth value when the set is empty, they are all undefined.

If one insists that the empty set is a subset of set A because there are no elements of the empty set that are not included in or are outside of set A then by that same line of reasoning every class which does not have actual members would also have to be subclasses of every other class, because from the standpoint of the actual world those classes would also be empty. For example, hobbits, and orcs, and Vulcans, and Klingons, and Jedi Knights would all be subclasses of the reptiles class. These classes only have members that are fictional so from the standpoint of the actual world they have to be considered empty and treated just like the empty set. In the actual world there is no member of the Klingons class that is outside of the reptiles class because there are no actual members of the Klingons class at all, therefore according to this reasoning the Klingons class must be a subclass of reptiles. The Klingons class would be a subclass of every class with actual members by the same reasoning. Every set that does not have actual members, and is thus empty relative to the actual world, would be a subset of every set that does have actual members.

The Klingons class would also be a subclass of ‘humans’. Wouldn’t it be incredibly confusing and seemingly quite inaccurate to say ‘All Klingons are humans’? But if ‘Klingons’ really is a subclass of humans then you should be able to say that and have it be correct, just as you could if you were to say ‘All Italian men are humans’; the latter is true because ‘Italian men’ is a legitimate subclass (species) of humans. Klingons are not a type of human, not in the actual world, and definitely not in Start Trek. So why would we want to classify them that way?

All of these classes would have to be subclasses of each other as well. Thus, one could say ‘All Klingons are Vulcans’ and ‘All Vulcans are Klingons’. This is a problem not only because both of these statements are false relative to the fictional Star Trek universe, where the ideas originated from, but how could they both simultaneously be a subclass of the other? These are self-contradictions.

‘All orcs are hobbits’ would not be true in Tolkien’s Middle-earth, but with this interpretation you would be committed to saying that it is true here in the real world, since it is the case that there are no orcs that exist in the actual world that are not hobbits; as would ‘All Klingons are orcs’ and every other universal claim about fictional beings and classes of fictional beings.

Classes with only fictional members would also be subclasses of class complements, which results in more self-contradictions. For instance, how could ‘All orcs are hobbits’ and ‘All orcs are non-hobbits’ both be true? But if ‘orcs’ is a subclass of both the hobbits class and of non-hobbits (how could that be?) then both categorical statements would have to be true, which is a blatant self-contradiction.

Not only does all of this result in some really ridiculous categorical statements that one is committed to saying are true, it also becomes an extremely bloated taxonomic system. What would be the point of creating subclasses within the humans class, for instance, which do not have any actual members that are humans? Why make a distinction and create a subclass if that subclass is empty?

Does the empty set contain elements that are prime numbers? No, obviously not, so why the hell would we say that it is a subset of prime numbers? It is not a species of prime numbers! It is not enough to point out that it does not have any members that are outside the set of prime numbers, if it is really a subset of prime numbers then it must also be true that all of its elements are also members of the prime numbers set, and that part is not fulfilled.

Mathematicians say that the empty set {} is the only set whose cardinality is zero. But think about that for a second: if a set is defined as a collection of things then how could there be a collection if there are no things?

Mathematicians tend to think of a set as a box. Not necessarily a physical box, but a container. The set {2, 4 ,6 ,8} would be like a box containing four numbers, 2, 4, 6, and 8. To them, the empty set would be like an empty box. However this actually leads to a self-contradiction. Here is how:

Hammack says that this set {{}} contains one element, itself. But how could that be? How would adding itself, a set without any elements, result in now having an element? I thought that it did not have an element! If the set that is added does not contain any members then there should be no difference at all between {}, {{}}, {{{}}}, {{{{}}}}, etc. The outer set of braces should be considered empty no matter how many empty sets there are as subsets because none of them have any members. If the empty set is equivalent to zero members then adding zero members does not add anything. {0}, {{0}}, {{{0}}}. It would be like if you had a set of zero books and you added that set to another set of zero books and then said that somehow the second set now had a book. How did he manage to pull that off? That is quite the little conjuring trick to make a member appear out of nothing. He is apparently quite a good magician, but to me it seems like some fuzzy accounting. The only way that it makes sense is if you think of the set as a container that is a separate object from its members. Then the set {{}} would be like an empty box inside of an empty box, which would mean that the outer box is not empty, as now it contains a box.

But consider this: If the empty set is a subset of every set then it must be a subset of itself, since it is a set. If you were to deny this I would ask why it is different from every other set. After all, it is the case that none of its members are outside of the empty set (because it has no members at all) so according to the reasoning used previously one would have to say that it is a subset of the empty set, or itself. But if the empty set has a member or subset (itself) then it is not empty. This is a self-contradiction. So is the empty set empty or not? Perhaps we could call this The Empty Set Paradox.

Now suppose that we have the set of non-empty sets; if the empty set is a subset of that one then the categorical statement ‘An empty set is a non-empty set’ or ‘The empty set is a non-empty set’ would have to be true because the empty set would be a species or type of non-empty set. But how could any set be a species of its opposite? An empty set, or the empty set, is not a type of non-empty set (its class complement) in the way that equilateral triangles are a specific type (or subset) of triangle. That is self-contradictory. Perhaps we could call this The Non-Empty Set Paradox.

It is obvious that mathematicians and the founders of Set Theory are thinking of sets as separate objects apart from their members, if they even have any members. Once again, not necessarily material objects, but containers; hence the box analogy. An empty box is still a thing, even if it is empty. So it seems that they want to make a distinction between the set and what it contains.

Gottlob Frege and Georg Cantor thought of sets this way. Bertrand Russell clearly did as well, that is obvious from the quote discussed earlier about ‘everything’ being ‘something’ and therefore a member of itself. Even the notion of a set being a member of itself is treating the set as a ‘thing’ that is separate from and in addition to its members, and then the question is whether this thing (the set) should be included in the set. We have also seen that John Venn thought of classes this way as well; the fact that Venn thought that universal categorical statements about empty classes would all be true, and even the fact he thought that empty classes would still exist even without any members indicates this. (One could make the same point about Frege, who also thought that there could be empty classes such as ‘the present king of France’ or ‘the greatest integer’.)

But I believe that this is wrong. The definition of a set is a group or collection of members. That means that if there is only one thing it is not a set, and obviously if there are no members then there is not a collection of members. Hence, if there are no members there is no set.

Empty classes should not even exist, except, perhaps, hypothetically. The empty set does not exist in any sense because not only is it not a collection of physical objects in the real world, it is not a collection of abstract objects, or fictional characters, or numbers, or anything else. It has absolutely no members of any kind, it is the set equivalent of a void, but that means that it is not really a set either. Sets are not separate things apart from their members, so a set of nothing is not a thing. The empty set is a collection of nothing, which is nothing.

Perhaps one could argue that even if it is a ‘non-set’ that would be something, therefore it would have to exist at least in some sense, but you cannot refer to it as ‘the empty set’ if you say that it is a non-set. That is self-contradictory. Obviously non-sets do exist, such as a single object that is not part of a collection, but the empty set does not. It is like a square-circle: We have a vague concept of it, but it does not really exist, even hypothetically, because that would be self-contradictory.

Hypothetical sets

In some cases one can imagine a hypothetical class but it does not really exist in the actual world any more than its members really exist in the actual world. It is not an actual class unless its members are actual.

This is an important distinction because in Set Theory the class is actual even if its members are not, it is just considered empty. For instance, 'the present king of the United States', for Frege, would be an actual class that is empty. But I see two problems with that: 1) it has no actual members (and hopefully it never will) and 2) even speaking theoretically it would only be a single thing, not a class.

Sets that have fictional members, such as orcs, hobbits, Klingons, etc., are fictional as well. They are not actual sets that are empty, therefore they are not actual subsets of actual sets such as the set of humans or the set of mammals.

We must keep the distinction between actual things and fictional things clear.[8] Let's imagine two piles, one of which has actual objects, and the other has things that are fictional or merely hypothetical. The one which is hypothetical has unicorns, dragons, mermaids, orcs, Achilles, etc. It would have fictional things, and also other hypothetical things. But there are no actual objects in this pile, so there is no actual pile of stuff; relative to the actual world there is nothing there. The objects are hypothetical therefore the pile is hypothetical. It makes absolutely no sense to create a separate category for these things in the actual objects pile if that category has no items in it. The category belongs in the pile of hypothetical objects where its members are and it could be a legitimate subclass and species.

The way to a better classification system

Frege and other contributors to Set Theory want to start with the classes and consider those to be the most fundamental units in their classification system. Having empty classes shows that the classes are considered to be more fundamental than the members. It also shows that the set is thought of as a separate 'thing' which still exists even if none of its members exist. Both of these ideas are wrong.

It is the *members* that are the most fundamental. We should start with the members and then figure out how to classify them in a way that makes sense.

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[8] There is also a distinction between possible and fictional. Assuming that imagining the fictional object or class to exist does not result in a self-contradiction, all fictional things would be logically possible but not all possible things would even be conceived of, so not all possible things would appear in works of fiction.

To consider the sets to be more fundamental than the members would be like saying that molecules are more fundamental than atoms.[9] (Part of the definition of 'molecule' is that it is a group of atoms bonded together.) If there are no atoms there is no molecule, and if there are no members there cannot be a group of members. The most fundamental units in the classification system should be the *elements*.

Richard Hammack says that all of mathematics can be described with sets. That seems to be the motivation for much of this; mathematicians and philosophers want a theory that can be used for anything and everything. The goal of Set Theory is not just to classify things, it is also to extend the logic that we use with sets to other areas besides what we ordinarily think of as sets, and be a more all-encompassing theory. But while they have done this I do not think that it has been done well. The theory is full of blatant self-contradictions. It is not even consistent with the definition of a set, which seems like a minimum requirement when you call it 'Set Theory'. It is like a horrible translation of a text from one language into another, and much is lost in translation. We should not try to apply the logic used with sets to situations that it was not meant for. Maybe you could use a saw to hammer in a nail if you really had to, but that is not really its purpose, and doing so is certainly less than ideal. It would be best to keep the logic related to sets confined to sets rather than trying to stretch it to apply to situations and claims that it was not designed for, and for which it does not really apply.

Over five hundred animal species are mentioned in Aristotle's writings. He seems to have been quite interested in trying to systematically classify them, which our current taxonomic system is based upon. It is fairly likely that Aristotle was the first person to attempt to do this, at least among the ancient Greeks. According to ancient accounts, Alexander of Macedon, commonly referred to as 'Alexander the Great', even sent back specimens to Aristotle of various plants and animals that he and his men discovered as Alexander conquered foreign lands. Alexander must have known that his former teacher was interested in collecting and studying these biological specimens, which would indicate that Aristotle must have been doing that for a good portion of his life.[10] One can imagine him trying to figure out how to classify these specimens relative to the plants and animals that he was already familiar with.

Suppose that you had a similar project. Let's imagine that you had the resources to build a large zoo and the capability to build many habitats and enclosures for many types of animals. The problem is that some of the animals that are being sent to you are ones that you are entirely unfamiliar with. You know little to nothing about them, but now you have to try to categorize

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[9] Or another comparison is that it is like treating compound statements as being more fundamental than simple statements in propositional logic.

[10] I have actually wondered if attempting to categorize all of the various animal species could have been what inspired Aristotle to develop categorical logic, along with his idea of defining things based upon genus, species, and difference, and perhaps some of his other ideas. If one imagines all of the logical possibilities for how a class (or a specimen) could be related to or not related to another class, it would yield the four categorical propositions (or for my version it would be three).

them in a way that makes sense. How would you do it? Would you try to come up with the classes first for every animal that you could possibly imagine and then see which if any of the animals that you actually have fit into those classes, or would you look for similar characteristics that the real animals have and then develop the classifications based upon your observations? I think the latter approach would work much better. It would be more accurate because you are starting with animals that you actually have, which is a more natural starting point than coming up with classes for things that may or may not exist. It is far more efficient because you would not be creating classes that will ultimately not be needed.

You would have to start with the most obvious shared characteristics. As your collection grew you would want and need to start making more refined distinctions, or in other words subclasses, and perhaps even subclasses of the subclasses, such as a subspecies of a certain species. If the collection was big enough eventually the organization of it (at least if you did a good job) would somewhat resemble the biological classification system of kingdom, phylum, class, order, family, genus, and species. It should look like an upside down tree (or, alternatively, we could think of it as the roots of a tree) in which it branches off into smaller but more numerous branches as it moves down. It goes from most general to most specific. That is exactly what we get using genus and species. A genus must always be more general than any of its species because the genus includes all members of that species along with the members of other species; each species has a difference from other species within that genus, which is why it is considered a separate species.

If we reach a level of specificity such that it has been narrowed down to a single unit or element, in this case a single animal, then obviously there could not be any more specific classification than that; there are no subclasses or species of an element. An element or member is the most specific that it is possible to get within that classification system, which is why it is the fundamental unit of that system (not sets); it is the end or limit for that particular root or branch of the tree. We start with all of the members, which is like the trunk of the tree, and then differences are noted until we get down to the level of each individual member. In many cases we would not want to get that specific with our classification, but you could.

None of the classes would be subclasses of themselves. If a class could be a subclass of itself then the species would not be more specific than the genus, it would just be an exact copy of the genus class. That would make no sense. There is no difference (not in the Aristotelian sense nor in the ordinary sense of the word) between the genus and species. Why would it be considered a separate class when it contains exactly the same members and excludes exactly the same members as the genus class? It is even referred to by the same name. There would be no reason to make that distinction in your classification system. Not only would it be needless redundancy, it also creates ambiguity and confusion.

The visitors who came to see the animals would be very disappointed if you created many enclosures and habitats for animals that were not there. Why build a container for it and list the animal as being included in the zoo's collection if it is not really there? We may suppose that this is not just a temporary situation, but a permanent one: the zoo never has had that animal and never will, yet some want to build an enclosure and environment for it and list it as one of the animals in the zoo's collection anyway. Why?

It would also be rather confusing if the sign said that there was a 'herd' of caribou, if there was actually only one of them, or a 'pack' of only one wolf; would it really be accurate to say that there was a pride of lions if that 'pride' had only one lion?

I think it would be good if mathematicians undertook a similar project to this using biological classification as a guide. I cannot do it because I do not think that I am familiar enough with mathematical entities and concepts to know how they ought to be organized based upon similar characteristics and differences. To do it right, it would probably need to be a group project for subject matter experts. If they started with the elements and then classified them in a way that was consistent and made sense, all would be well. There could be some disagreement over how certain things should be classified, but after some haggling they could probably come up with a pretty good system that is at least fairly consistent. That would be a much better approach than trying to start with the classes.

I realize that Math is more abstract than Biology, but that is all the more reason why we need to stay tethered to reality. If we do not it is easy to get really far off track.

Having a classification system like this, rather than Set Theory, would help to make math easier to understand for students because it would get rid of some of the most bizarre, counterintuitive, and outright self-contradictory claims. (No need to fear running out though, there are still plenty left.) Students often blame themselves, thinking that the reason that it does not make sense to them is because they just cannot do math, but in some cases their intuitions are not wrong, the theory is wrong.

There can be a certain elegance to simplicity. I think academics are drawn to complicated systems but simpler is often better and closer to the truth. Let's use Ockham's razor to shave off what we do not need.

To summarize what has been said in this final section, here is a list of rules regarding classes and class formation:

Rules

1) A class is defined as a collection or group of things; therefore a class must have at least two members. Single items should not be treated as a class.
2) Members are the most fundamental units in the classification system; classes and subclasses are merely different ways of organizing and grouping members.

3) Members/elements are singular, classes/sets are plural; it follows that no class can simultaneously be both the class (a group of members) and a member of the same class.
4) A class can be considered a member of some other class only in the circumstance in which it is treated as a single unit of the other class, such as when a team is considered a unit of a league, or a corporation or government is thought of as a single entity rather than as a collection of members. The class must be treated either as a unit or as a collection of units, it cannot be both simultaneously within the same context. (See Rule 3.)
5) If a class is described as having a predicate it is the members of the class that have it, not the class itself. The class 'dogs' is not itself a mammal, though all of its members are, but it is not correct to say that it is a 'non-mammal' either; the class is not a separate object that has (or lacks) its own separate predicates apart from the predicates of its members.
6) If two sets have every element in common, meaning that they include exactly the same elements and exclude all the same elements, they are actually the same set. The set may be referred to by different names, such as with 'domesticated dogs' and 'Canis lupus familiaris', but since a set is defined as a collection of things, if the things are all the same then it is the same collection, and thus the same set.
7) If and only if every element of A is also an element of B and some but not all elements of B are also elements of A then A is a subset of B.
8) Sets can sometimes also be considered subsets for a more general classification (a species or type of some more broadly defined genus) but no set can be a subset of itself, as this would result in the self-contradictory claim that a part is equal to the whole.
9) Because a class is defined as a collection or group of members there can be no empty classes. If there are no members there is no collection of members. If the members are actual and the characteristics by which they are grouped are actual then the set is actual; if the members of the class are fictional or merely hypothetical then the class is merely fictional and/or hypothetical; if the members are neither then there is no class at all, such as with the empty set, which is not really a set.